Indian Statistical Institute, Bangalore Centre.
FINAL Exam : Topology I

Instructor : Yogeshwaran D.

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Max. marks - 50 marks. **Time Limit : 3 hours.**

- **Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention the result clearly. Citing assignment problems without suitable explanations will not be accepted.**
- **Every question carries 10 points. Parts within a question might not carry equal weightage.**
- **Answer any five questions. At least 2 questions each from both Part A and Part B are to be answered.**
- **Only the first five answers will be evaluated.**

1 PART A

1. Let X be a T_1 space. Let $\{f_\alpha\}_{\alpha \in J}$ be family of $[0,1]$-valued continuous functions on X such that for each x and any open neighbourhood U of x, there exists a function f_α such that $f_\alpha(x) > 0$ and $f_\alpha \equiv 0$ on U^c. Show that the following function

$$F : X \to [0,1]^J, \ F(x) = (f_\alpha(x))_{\alpha \in J},$$

is well-defined and an imbedding of X into $[0,1]^J$. Is the converse true ? In other words, does the existence of the above imbedding F guarantee the existence of a class of "seperating" functions $\{f_\alpha\}$ as above ?

2. Is the Stone-Cěch compactification of S_Ω equivalent to the one-point compactification ? Prove or disprove. Does the statement also hold for other compactifications of S_Ω ?

3. Show that at most countable product of sequentially compact spaces is sequentially compact. Is $\{0,1\}^{[0,1]}$ sequentially compact ?

4. STRONG FORM OF URYSOHN LEMMA : Let X be a normal space. Show the following statement : There exists a continuous function $f : X \to [0,1]$ such that $f \equiv 0$ on A, $f \equiv 1$ on B and $0 < f(x) < 1$ for $x \notin A \cup B$ iff A and B are disjoint closed G_δ sets in X. Recall that a set A is called a G_δ set if it is the intersection of a countable collection of open sets.

5. Which of the following properties does $\mathbb{R}_K$ satisfy - Connectedness, Path connectedness, Hausdorff, Regular and Heine-Borel property (i.e., all closed and bounded subsets are compact) ?

2 PART B

6. Given a space X, the cone of X is defined as the following quotient space - $C(X) := X \times I / X \times \{1\}$. Show the following :

   (a) Show that $C(X)$ is always contractible.

   (b) If $X \subset \mathbb{R}^d$ for some $d \geq 1$, then $C(X)$ is homeomorphic to the following subpace of $\mathbb{R}^{d+1}$ - $GC(X) := \{t(x,0) + (1-t)e_{d+1} : x \in X, t \in I\}$ where $e_{d+1} = (0,\ldots,1) \in \mathbb{R}^{d+1}$.

   (c) Show that $C(X)$ is homeomorphic to the closed unit ball when $X = \mathbb{S}^n$, $n \geq 0$.

7. Prove that the map $f : \mathbb{S}^1 \to \mathbb{D}^2 \times \mathbb{S}^1$ given by $f(z) = (z,z)$ is not homotopic to a constant map and the map $f_n(z) = z^n : \mathbb{S}^1 \to \mathbb{S}^1$ is a covering map. Also, what is the induced homomorphism $(f_n)_*$ between the fundamental groups ?
   NOTE : We view $\mathbb{S}^1, \mathbb{D}^2$ as subspaces of the complex plane and the notation is to be suitably interpreted.

8. Compute fundamental group of $\mathbb{R}^d - \{p,q\}$ where p,q are distinct points in $\mathbb{R}^d$ and $d \geq 1$ but $d \neq 2$.

9. Let $p : E \to B$ be a covering map. Prove the following claims.

   (a) Let B be connected. If $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. If $k = 1$, what can you say about p ?

   (b) If $q : B \to C$ is also a covering map and $q^{-1}(c)$ is finite for all $c \in C$, then $q \circ p$ is a covering map.